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[illegible]

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II. MINIMAL COUPLING FORMALISM

Consider a quantum particle with wave function $\Psi(\mathbf{L}, t) = \Psi_1(\mathbf{L}, t) + i\Psi_2(\mathbf{L}, t)$, Ψ_1 and Ψ_2 being real, that satisfies the time-dependent Schrödinger equation $i\hbar\,\partial_t\Psi = \hat{H}\Psi$. This

Case 1: Spatial derivatives of P are negligible, and ML/B and PL^2/B are both $\ll 1$ so that we may retain terms only to linear order in these quantities in the nondimensionalized equations of motion (lengths measured in units of L, time in units of $L^2\sqrt{\ /B}$). On the quantum side, this case corresponds to the semiclassical approximation with weak potentials. On the elastic side, it corresponds to small terminal twist angle $= (1+\nu)ML/B$ where ν is Poisson's ratio, and P well below the Euler buckling threshold $^2B/L^2$. Equations (6) reduce to

$$-\ \ddot{\Psi}_1 = B\Psi_1'''' + M\Psi_2''' + P\Psi_1'', \tag{7a}$$

$$-\ \ddot{\Psi}_2 = B\Psi_2'''' - M\Psi_1''' + P\Psi_2''. \tag{7b}$$

To linear order in ML/B, the twist-induced tension contribution to P is not accounted for, and all elastic parameters become independent of one another. Incidentally, *static* heli-

not in a thermal equilibrium sense since here the total energy (found by changing the signs of the last three terms in Eq. (12) and integrating over x) is not also periodic in .

Letting Eqs. (14) describe the periodic straight rod and setting $M = P = 0$ yields a set of de Broglie relations: $p_x = \int dx\, g = \hbar k_n$ (using one of the first two methods for g) and $E = \hbar\omega_n$ (equal parts kinetic and potential energy and also apmore exotic physics, such as the Zak phase [32], but in a classical setting.

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APPENDI A: ENERG -MOMENTUM TENSOR

The energy-momentum tensor, also known as the stress-energy tensor, is usually derived assuming the Lagrangian density depends on field variable derivatives only up to first order. Here, we derive it (in the notation of Ref. [33]) for a Lagrangian density, such as Eq. (12), that depends on field variable $u(x)$ derivatives up to second order: $\mathcal{L} = \mathcal{L}(u, \nabla u, \nabla\nabla u, x)$. The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial u_i} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial u_{i,\alpha}} + \partial_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial u_{i,\alpha\beta}} = 0. \tag{A1}$$

Under an ϵ-family of transformations of the field variables, and writing $\delta \equiv d/d\epsilon$ at $\epsilon = 0$,

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial u_i}\delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,\alpha}}\delta u_{i,\alpha} + \frac{\partial \mathcal{L}}{\partial u_{i,\alpha\beta}}\delta u_{i,\alpha\beta}. \tag{A2}$$

Combining the previous two equations yields

$$\delta\mathcal{L} = \partial_\alpha \left[\left\{ \frac{\partial \mathcal{L}}{\partial u_{i,\alpha}} - \partial_\beta \left(\frac{\partial \mathcal{L}}{\partial u_{i,\alpha\beta}} \right) \right\} \delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,\alpha\beta}} \right.$$